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LETTER TO THE EDITOR

Visible points in a lattice

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Abstract. The geometrical structure associated with visible points in a lattice is investigated. It appears to be the Möbius transform of the original lattice. The related symmetry group and structure factor are explicitly derived.

Let Λ be an n-dimensional lattice of points. I call F_Λ the set of points visible from the origin (excluding the point at the origin). For example, if $\Lambda = \mathbb{Z}^2$, this set (figure 1) is equivalent to that of rational irreducible fractions. Or, if one restrict to the values $0 \leqslant x, y \leqslant d$, it is equivalent to the Farey series of order d. The aim of this letter is to present some properties of this set of points. More precisely, I demonstrate the following three properties:

(a) the set F_Λ can be considered as the Möbius transform of Λ;

(b) F_Λ is invariant under the action of $\mathrm{SL}(n, \mathbb{Z})$;

(c) the structure factor of F_Λ can be calculated.

In the following I shall mainly refer to the simplest case $\Lambda = \mathbb{Z}^2$, but most of the reasoning and results are easily extended in general dimensions.

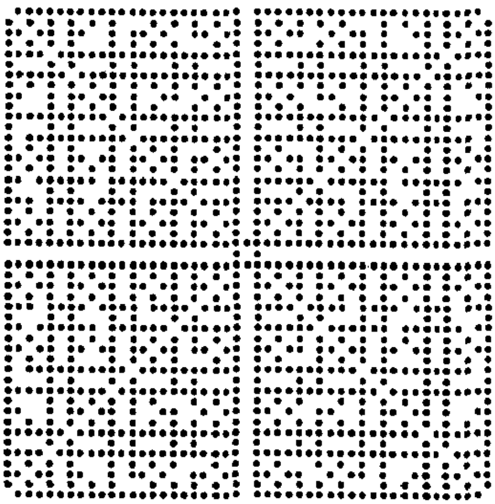

Figure 1. A limited view of the visible point in the case $\Lambda = \mathbf{Z}^2$. The full set is invariant under $\mathrm{SL}(2, \mathbf{Z})$.

Consider the function $h_\Lambda(x, y)$ which takes the value 1 if $(x, y) \in \Lambda$ and zero in the opposite case. Note that in [1] the author gives the value -1 to the function where it is zero here. In the following we note F for $F_{\mathbb{Z}^2}$. and $2F$ for $F_{(2\mathbb{Z})^2}$, etc...

If (x, y) is a visible point of $\mathbb{Z}^2$ then (px, py), $p \in \mathbb{N}^*$, is a visible point of $(p\mathbb{Z})^2$, so

$$\mathbb{Z}^2 = \cup (pF) \tag{1}$$

and

$$h_{\mathbb{Z}^2} = h_F + h_{2F} + h_{3F} + \ldots = \sum_{p \in \mathbb{N}^*} h_{pF} \tag{2}$$

It is clear that the two sets pF and qF do not intersect when $p \neq q$. As a consequence h_F is the Möbius transform of $h_{\mathbb{Z}^2}$ (cf Hardy and Wright [2, theorem 270])

$$h_F = \sum_{l \in \mathbb{N}^*} \mu(l) h_{(l\mathbb{Z})^2} \tag{3}$$

where $\mu(l)$ is the Möbius function:

$$\mu(l) = \begin{cases} 1 & \\ 0 & \text{if 1 contains a squared term in its prime factor decomposition} \\ (-1)^p & \text{if } l \text{ contains } p \text{ different prime factors.} \end{cases} \tag{4}$$

All of this is true in general dimensions.

F is clearly not periodic. One simple proof consists in calculating the frequency of visible points. If it is irrational, then it will not be possible to construct a repetitive cell (even very large) associated with a periodic structure. Let p be this quantity:

$$\begin{aligned} p &= \frac{\sum_{r \in \mathbb{Z}^2} h_F}{\sum_{r \in \mathbb{Z}^2} h_{\mathbb{Z}^2}} \\ &= \frac{\sum_{r \in \mathbb{Z}^2} \sum_{l \in \mathbb{N}^*} \mu(l) h_{(l\mathbb{Z})^2}}{\sum_{r \in \mathbb{Z}^2} h_{\mathbb{Z}^2}} \\ &= \sum_l \mu(l) \frac{\sum_{r \in \mathbb{Z}^2} h_{(l\mathbb{Z})^2}}{\sum_{r \in \mathbb{Z}^2} h_{\mathbb{Z}^2}} = \sum_l \mu(l) \frac{1}{l^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \end{aligned} \tag{5}$$

In d dimensions the non-periodicity of F is a consequence of the (yet unproved for odd $d > 3$) irrationality of the Riemann function $\zeta(d)$. Note that the converse is not true: it is easy to find other proofs for the non-periodicity of F, but, unfortunately, this does not imply the irrationality of $\zeta(d)$.

F is not periodic, but it presents nevertheless some regularities, which are already perceived by looking at figure 1. In particular, prime numbers do structure this set; square 'blocks' whose side equals a prime number form *patterns* which seem to repeat periodically, but with some mistakes. Without being able to be more precise, I would say that these blocs play a role similar to that of periodic approximants in a quasicrystal. But here it is prime numbers instead of irrational numbers which come into play. In fact I show below that this set is almost periodic.

F has D_4 dihedral group symmetry. In addition, and much more interesting, F is also invariant under the action of SL(2, $\mathbb{Z}$). As a proof, it suffices to remark that any element M of SL(2, $\mathbb{Z}$) transforms a unit cell of $\mathbb{Z}^2$ (a unit area parallelogram) into another unit cell. The basis vectors defining a unit cell connect the origin to a visible point of $\mathbb{Z}^2$. Therefore M brings a visible point onto a visible point and let F be invariant. As all the $\mathbb{Z}^2$ unit cells can be put in correspondance under the action of

SL(2, $\mathbb{Z}$), and any visible point belongs to a unit cell (in fact to infinity), one deduces that F possess this symmetry. Moreover the group action is transitive (which is not the case for its action onto $\mathbb{Z}^2$ for example).

In fact, in any dimension, the sets pF (equation (1)) are the equivalence classes of the lattice under SL(n, $\mathbb{Z}$).

Let $S_F(q)$ be the structure factor of the set F, defined as follows:

$$S_F(q)=\frac{1}{C}\sum_{r\in F}\mathrm{e}^{\mathrm{i}qr} \tag{6}$$

where C is a normalization factor which we take such that $S_{\mathbb{Z}^2}(q)=1$ at the reciprocal network nodes of $\mathbb{Z}^2$. Using the above results, it becomes simply

$$\begin{aligned} S_F(q) &= \frac{1}{C}\sum_{r\in\mathbb{Z}^2} h_F(r)\,\mathrm{e}^{\mathrm{i}qr}=\frac{1}{C}\sum_{r\in\mathbb{Z}^2}\sum_{l\in\mathbb{N}^*}\mu(l)h_{(l\mathbb{Z})^2}(r)\,\mathrm{e}^{\mathrm{i}qr} \\ &= \frac{1}{C}\sum_{l\in\mathbb{N}^*}\mu(l)\sum_{r\in\mathbb{Z}^2}h_{(l\mathbb{Z})^2}(r)\,\mathrm{e}^{\mathrm{i}qr} \\ &= \sum_{l\in\mathbb{N}^*}\mu(l)\frac{1}{l^2}S_{\mathbb{Z}^2}(lq). \end{aligned} \tag{7}$$

But $S_{\mathbb{Z}^2}(lq)=1$ if and only if

$$lq=2\pi m_x\boldsymbol{x}+2\pi m_y\boldsymbol{y} \qquad m_x, m_y\in\mathbb{N} \tag{8}$$

and $S_{\mathbb{Z}^2}(lq)=0$ in the opposite case. If q reads

$$q=2\pi\frac{q_x}{n_x}+2\pi\frac{q_y}{n_y} \qquad \text{with} \quad q_x\wedge n_x=1,\ q_y\wedge n_y=1 \tag{9}$$

then

$$S_F\left(2\pi\frac{q_x}{n_x}, 2\pi\frac{q_y}{n_y}\right)=\sum_{p\in\mathbb{N}^*}\frac{\mu(pk)}{(pk)^2} \qquad \text{with } k=\gcd(n_x, n_y). \tag{10}$$

The structure factor of F is shown in figure 2. The area of the circles is proportional to the absolute value of $S(q)$. The largest circles are centred on the nodes of the reciprocal network of $\mathbb{Z}^2$. $S_F(q)$ is invariant under SL(2, $\mathbb{Z}$). This is easily seen by using the transitive property of the group onto F. If M is a matrix of SL(2, $\mathbb{Z}$), then

$$\begin{aligned} S_F(q) &= \frac{1}{C}\sum_{r\in F}\mathrm{e}^{\mathrm{i}qr}=\frac{1}{C}\sum_{r\in F}\mathrm{e}^{\mathrm{i}qMr} \\ &= \frac{1}{C}\sum_{r\in F}\mathrm{e}^{\mathrm{i}M^tqr}=S_F(M^tq). \end{aligned} \tag{11}$$

The set F is a geometrical model which illustrates some quantities and operations of number theory, associated in particular with prime numbers and the modular group.

Let me repeat that the above described properties generalize without difficulty to any lattice in any dimension. Indeed, on one hand, in a given dimension, all lattices are identical modulo an affine transformation. On the other hand, the visibility property is itself an affine (and even projective) property. It remains to take into account the

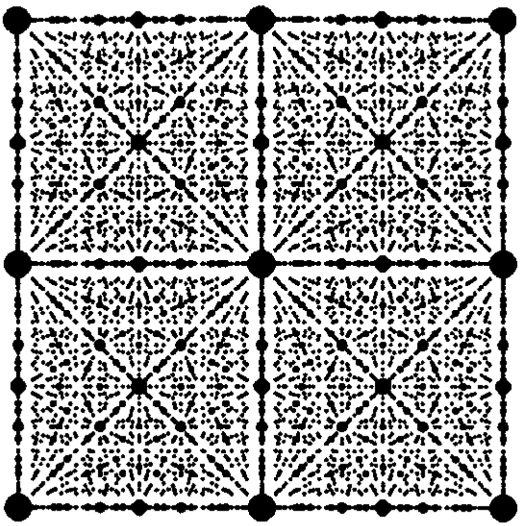

Figure 2. The structure factor of the set represented in figure 1. The true function is dense; only the largest peaks are shown. The circle areas are proportional to the absolute value of $S(q)$. The structure factor is invariant under $SL(2, \mathbb{Z})$.

relative densities of the lattices in d dimensions. For example the factor l^2 which appears in the denominator in $S_F(q)$ (equation (7)) becomes l^d. The lattice metrics only matter if one wants to precisely localize the points of F or the vectors q in the reciprocal space.

Can F be of interest in physics? Here are some points to consider.

As visible points of a lattice, the points in F could play a role in elucidating the geometrical aspects of channelling experiments.

In the field of quasicrystals, a well known method to produce quasiperiodic tilings consists of the selection of points in a high-dimensional lattice and their projection on a so-called 'physical space' of lower dimension. If the tiling has certain self-similar properties, the physical space is the eigenspace of a suitable hyperbolic matrix in $SL(n, \mathbb{Z})$. The set of iterates of this matrix produce new points of the tiling. This 'inflation mapping' [3] also generates the successive periodic 'approximants' of the quasicrystal. Now it is clear from what is said above that if one starts with a visible point of the high-dimensional lattice, the iterates under the inflation map will also be visible points. This is a sort of hidden symmetry in the quasicrystal, whose points can be naturally split into sets of points which, before projection, belong to the same pF. Note however that this inflation map is of a semigroup nature.

If one considers the Voronoi cell (Wigner–Seitz or Brillouin zone) which surrounds the origin and closes it onto itself (into a torus), then each point in F is in one-to-one correspondence with a periodic orbit on the torus. If one tries to visit this zone in the most homogeneous manner one should follow a line directed towards a far point in F. If one knew how to extend the present work to the case of visible points in regular tilings of the hyperbolic plane, this could prove interesting for certain problems related to chaos where one enumerates closed geodesics in certain hyperbolic manifolds.

More exotically, one can consider the Olbers paradox of astrophysics by supposing the stars located on a network and calculating the light intensity at the origin as a function of the solid angle. Only the F points contribute.

This work started from a discussion about Farey sequences with Clément Sire. After it was completed, Marjorie Sénéchal has brought to my attention a related work by M Schroeder which studies this set in the case $\Lambda = \mathbb{Z}^2$ and calculates its Fourier transform on a computer. The above described results are more general and simplify greatly the analysis of this set.

References

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